Supplemental Information for "Elections, Protest, and Alternation of Power"

Andrew T. Little* Joshua A. Tucker† Tom LaGatta‡

May 2015

In this supplemental appendix, we first give derivations of the citizens beliefs and then the equilibrium condition. Next, we provide proofs of the propositions in the main text.

Conditional distributions of ω

Here we derive the most complex posterior belief about ω: the conditional belief given e, m, and θ_j. The other posteriors follow from a similar calculation. By the assumptions in the main text, the joint distribution of ω, e, m, and θ_j is a multivariate normal with mean vector $(\mu_0, \mu_0 + \overline{x}, \overline{x}, \mu_0)$ and covariance matrix:

$$\Sigma = \begin{array}{c} \\ \omega \\ e \\ m \\ \theta_j \end{array} \begin{array}{c} \begin{array}{cccc} \omega & e & m & \theta_j \end{array} \\ \begin{pmatrix} \tau_0^{-1} & \tau_0^{-1} & 0 & \tau_0^{-1} \\ \tau_0^{-1} & \tau_0^{-1} + \tau_e^{-1} & \tau_e^{-1} & \tau_0^{-1} \\ 0 & \tau_e^{-1} & \tau_e^{-1} + \tau_m^{-1} & 0 \\ \tau_0^{-1} & \tau_0^{-1} & 0 & \tau_0^{-1} + \tau_\theta^2 \end{pmatrix} \end{array} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

*Assistant Professor, Department of Government, Cornell University, andrew.little@cornell.edu.
†Professor, Department of Politics, New York University, joshua.tucker@nyu.edu.
‡Courant Instructor / PIRE Fellow, Courant Institute of Mathematical Sciences, New York University, lagatta@cims.nyu.edu

where $\Sigma_{11} = \tau_0^{-1}$ (which uniquely determines the remainder of the partition). As a consequence, the desired posterior is normal (see Greene 2008, p. 1014, or LaGatta 2013 for the general case). The conditional mean is

$$\begin{aligned}\overline{\overline{\mu}}(e, m, \theta_j) &= \mu_0 + \Sigma_{12}\Sigma_{22}^{-1}(e - (\mu_0 - x), m - \overline{x}, \theta_j - \mu_0) \\ &= \frac{\tau_0\mu_0 + (\tau_e + \tau_m)(e - \overline{x}) + \tau_\theta\theta_j - \tau_m(m - \overline{x})}{\tau_0 + \tau_e + \tau_m + \tau_\theta}\end{aligned}$$

and the conditional precision is given using the Schur complement:

$$(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1/2} = \tau_0 + \tau_e + \tau_m + \tau_\theta.$$

Derivation of the Citizen Equilibrium Strategy

For a fixed ω and cutoff rule $\hat{\theta}(e)$, the probability that a citizen protests is:

$$Pr\Big(\omega + \nu_j < \hat{\theta}(e)\Big) = \Phi\left(\tau_\theta^{1/2}(\hat{\theta}(e) - \omega)\right) \quad (1)$$

By the continuum assumption, the size of protests is exactly the right-hand side of (1). Hence the protest size is continuous and increasing in ω, approaches 0 as $\omega \to -\infty$ and approaches 1 as $\omega \to \infty$. So for each e there will be a critical $\tilde{\omega}(e)$ such that the protest induces the incumbent to step down if and only if $\omega < \tilde{\omega}(e)$, given by:

$$\begin{aligned}\Phi\left(\tau_\theta^{1/2}(\hat{\theta}(e) - \tilde{\omega}(e))\right) &= \rho^* \\ \tilde{\omega}(e) &= \hat{\theta}(e) - \tau_\theta^{-1/2}\Phi^{-1}(\rho^*)\end{aligned}$$

where ρ^* is derived in the main text. The posterior belief about ω held by a citizen observing

$\theta_j = \hat{\theta}(e)$ and e is normally distributed with mean:

$$\overline{\mu}(e, \hat{\theta}(e)) = \frac{\tau_0\mu_0 + \tau_e e + \tau_\theta \hat{\theta}(e)}{\tau_0 + \tau_e + \tau_\theta}$$

and precision $\tau_0 + \tau_e + \tau_\theta$. So the probability this citizen assigns to the protest succeeding is:

$$\begin{aligned} Pr(\omega < \tilde{\omega}(e)) &= \Phi((\tau_0 + \tau_e + \tau_\theta)^{1/2}(\tilde{\omega}(e) - \overline{\mu}(e, \hat{\theta}(e)))) \\ &= \Phi\left((\tau_0 + \tau_e + \tau_\theta)^{1/2}\left(\hat{\theta}(e) - \tau_\theta^{-1/2}\Phi^{-1}(\rho^*) - \frac{\tau_0\mu_0 + \tau_e e + \tau_\theta\hat{\theta}(e)}{\tau_0 + \tau_e + \tau_\theta}\right)\right) \\ &= \Phi\left(\frac{\hat{\theta}(e) - \mu_{RHS}(e)}{\sigma_{RHS}}\right), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \mu_{RHS}(e) &= \frac{\tau_0\mu_0 + \tau_e e + (\tau_0 + \tau_e + \tau_\theta)\tau_\theta^{1/2}\Phi^{-1}(\rho^*)}{\tau_0 + \tau_e} \\ \sigma_{RHS} &= (\tau_0 + \tau_e + \tau_\theta)^{1/2}(\tau_0 + \tau_e)^{-1}. \end{aligned}$$

For the derivations in the appendix, we write the equilibrium condition for the citizen threshold as a $\hat{\theta}(e)$ that satisfies the following equation:

$$\Phi\left(\frac{\hat{\theta}(e) - \mu_{RHS}(e)}{\sigma_{RHS}}\right) = \frac{c + b_1\hat{\theta}(e)}{-\hat{\theta}(e)(b_3 - b_1 - b_2)}, \tag{3}$$

which is equivalent to the equation 3 in the main text.

Proof of Lemma 1

First, note that $\tau_{RHS} = \sigma_{RHS}^{-2}$ is strictly increasing in $\tau_0 + \tau_e$ and approaches 0 as $\tau_0 + \tau_e \to 0$. Second, by standard properties of normal a normal CDF, the left-hand side of the equilibrium condition is equal to 1/2 and attains it's maximum slope of $\sqrt{\tau_{RHS}/2\pi}$ at

$\hat{\theta}(e) = \mu_{RHS}$. Third, μ_{RHS} is an increasing affine function of e.

Let $\underline{d} > 0$ be minimum slope of the right-hand side of the equilibrium condition on $[\underline{\theta}, \overline{\theta}]$. Let θ_5 be the value of $\hat{\theta}(e)$ such that the right-hand side is equal to 1/2, and let and d_5 be the slope of the right-hand side at θ_5.

The above arguments imply that there exists a τ^* such that if $\tau_0 + \tau_e < \tau^*$ then $\sqrt{\tau_{RHS}/2\pi} < d_5$, which guarantees a unique solution to equation 3 and hence unique equilibrium.

If τ_e is high enough that $\sqrt{\tau_{RHS}/2\pi} > d_5$ – which may hold if $\tau_e = 0$, but will always require $\tau_e > 0$ if τ_0 is sufficiently small – then when e is such that $\mu_{RHS} = \theta_5$ there will be an intersection at $\hat{\theta}(e) = \theta_5$ where the left-hand side is increasing faster than the right-hand side. Further, there will be an open interval $(\underline{\mu}, \overline{\mu})$ and a corresponding $(\underline{e}, \overline{e})$ such that there is a $\hat{\theta}(e)$ "near" μ_{RHS} where the equilibrium condition is met and the left-hand side is increasing faster than the right-hand side. Since the left-hand side is above the right-hand side at $\underline{\theta}$ and vice versa at $\overline{\theta}$, this implies there must be two additional intersections, and hence multiple equilibria.

Proof of Proposition 1

For this and later proofs it is useful to write the equilibrium condition for the citizen threshold as a $\hat{\theta}(e)$ solving:

$$\Phi\left(\frac{\hat{\theta}(e) - \mu_{RHS}(e)}{\sigma_{RHS}}\right) - \frac{c + b_1\hat{\theta}(e)}{-\hat{\theta}(e)(b_3 - b_1 - b_2)} \equiv G(\hat{\theta}(e); e) = 0$$

First we need to determine how the equilibrium threshold changes as a function of c and e. Implicitly differentiating with respect to c gives:

$$\frac{\partial\hat{\theta}(e)}{\partial c} = \frac{1/(\hat{\theta}(e)(b_3 - b_1 - b_2))}{\frac{\partial G}{\partial \theta}}$$

The numerator is positive. $G(\underline{\theta}) > 0$ and $G(\overline{\theta}) < 0$ and both sides are continuous in e and θ, so if the intersection is unique, then $\frac{\partial G}{\partial \theta}\big|_{\theta=\hat{\theta}(e)} < 0$. Similarly, the right-hand side is also increasing faster than the left-hand side at the lowest and highest intersection,[1] and hence the expression is negative for these equilibria as well. So if there is a unique equilibrium, the threshold is decreasing (and hence fewer citizens protest) as c increases, and if there are multiple equilibria this comparative static holds in the equilibrium with the highest and lowest protest level. The protest threshold is decreasing in e in these two cases by a similar argument.

For a fixed ω and e, the protest size is:

$$\rho(\omega, e) = \Phi(\tau_\theta^{1/2}(\omega - \hat{\theta}(e)))$$

Conditional on e, ω is normally distributed with mean $\overline{\mu}(e)$ and precision $\tau_0 + \tau_e$, and hence ρ is a normal random variable with CDF:

$$Pr(\rho < x|e) = \Phi(\tau_\rho^{1/2}(\overline{\mu}(e) - x)) \tag{4}$$

where $\tau_\rho = \frac{\tau_\theta(\tau_0+\tau_e)}{\tau_\theta+\tau_0+\tau_e}$. Differentiating with respect c gives that this expression has the same sign as $\frac{\partial \hat{\theta}(e)}{\partial c}$, which as shown above is negative in the unique equilibrium case and the equilibria with the most and least protest in the multiple equilibrium case. So, for any $c_1 < c_2$, the distribution of ρ under c_1 first order stochastic dominates the distribution of ρ under c_2. Evaluating at $x = \rho^*$ gives the likelihood of protest succeeding is decreasing in c, the first order stochastic dominance property implies the expected level of protest is decreasing in the cost of protest as well. An analogous argument applies when differentiating with respect to e.

[1]For a set of measure zero election results the slope of the left-hand side and right-hand side is equal at the intersection and this derivative is undefined.

Discussion of Monotonicity and Lowest/Highest Equilibrium Restriction

Following a standard restriction in the global games literature, in the multiple equilibria case we only consider equilibria where $\frac{\partial G}{\partial \theta}\big|_{\theta=\hat{\theta}(e)} < 0$. This condition does not hold for the "middle" equilibrium when there are three intersections, which are unstable for reasons analogous to the mixed strategy equilibrium in a complete coordination game.

Since the protest threshold is decreasing for some election results, in any monotone equilibrium the threshold must be decreasing in the election result everywhere. In the case where there are at most 3 $\hat{\theta}(e)$'s, the middle equilibrium must have $\frac{\partial G}{\partial \theta}\big|_{\theta=\hat{\theta}(e)} > 0$, so all monotone equilibria meeting the requirement that this derivative be negative are of the form "highest protest equilibrium if $e < e^*$ and lowest protest equilibrium is $e > e^*$". When there are 5 or more equilibria it would be possible to have a monotone equilibrium where an protest threshold other than the highest or lowest possible is played, but substantial numerical analysis indicates there are never more than 3 equilibrium thresholds.

An obvious drawback to an argument based on equilibrium multiplicity is that other equilibria could be played as well, some of which do not involve complying with democratic rules. However, compliance with a majoritarian electoral rule and not some other electoral threshold is particularly appealing for various reasons. Following an electoral rule – particularly one at a natural threshold like a majority vote share – is a natural focal point in the sense coined by Schelling (1960, ch. 3). Citizens could threaten to protest against leaders who don't achieve the electoral threshold plus five percent, or on any publicly observed "sunspot", but these simply seem less natural than coordinating against law breakers. Further, such a rule may be optimal by putting popular leaders in office peacefully.

Proof of proposition 2

The payoff to standing firm at the first opportunity as a function of the realization of ρ is:

$$u^I_{SF}(\rho, e) = \begin{cases} 1-\rho & \rho < \rho^* \\ y - \gamma\rho & \rho > \rho^* \end{cases}$$

Which is strictly decreasing in ρ. Since for any $e_2 > e_1$, the distribution of ρ under e_1 first order stochastic dominates the distribution under e_2, the expected payoff for standing firm is strictly increasing in e. The limiting payoffs follow from the fact that $Pr(\rho < \rho^*|e)$ approaches 1 and $\mathbb{E}[\rho|\rho < \rho^*|e]$ approaches 0 as $e \to \infty$ and $\Pr(\rho > \rho^*|e)$ approaches 1 and $\mathbb{E}[\rho|\rho > \rho^*, e]$ approaches 1 as $e \to -\infty$. Part ii follows immediately from part i.

Proof of proposition 3

When there is a unique equilibrium in the protest stage for all e, then implicitly differentiating the equilibrium condition, $\frac{\partial \hat{\theta}}{\partial e}$ is continuous and strictly decreasing in e (see the proof of proposition 1). So, the incumbent utility for standing firm after the election and making the second optimal choice to step down is continuous and increasing in e. Combining with proposition 2 implies that there is a unique e^* such that the incumbent steps down if $e < e^*$, stands firm if $e > e^*$, and can stand firm or step down if $e = e^*$. So, e^* is the unique enforceable electoral rule. Since $e_r = e^* + \epsilon$ is not enforceable for any $\epsilon > 0$, e^* is not strongly enforceable, completing part i.

For part ii, since $\underline{u}^I_{SF}(e) < \overline{u}^I_{SF}(e)$ and both are continuous and strictly increasing on $(\underline{\underline{e}}, \overline{\overline{e}})$, $\underline{u}^I_{SF}(\underline{\underline{e}}) < y$ implies either $\underline{u}^I_{SF}(e) < y$ for all $e \in (\underline{\underline{e}}, \overline{\overline{e}})$, in which case define $\underline{e} = \underline{\underline{e}}$ or there exists a $\hat{e}$ such that $\underline{u}^I_{SF}(\hat{e}) = y$, in which case define $\underline{e} = \hat{e}$. Similarly, $\overline{u}^I_{SF}(\overline{\overline{e}}) > y$ implies either $\overline{u}^I_{SF}(e) > y$ for all $e \in (\underline{\underline{e}}, \overline{\overline{e}})$, in which case define $\overline{e} = \overline{\overline{e}}$ or there exists a $\hat{e}$

such that $\overline{u}^I_{SF}(\hat{e}) = y$, in which case define $\overline{e} = \hat{e}$. So, $\underline{u}^I_{SF}(e) < y < \overline{u}^I_{SF}(e)$ for all $e \in (\underline{e}, \overline{e})$. (It must be the case that $\underline{e} < \overline{e}$ since $\underline{u}^I_{SF}(e) < \overline{u}^I_{SF}(e)$.) Since $(\underline{e}, \overline{e})$ is an open interval, for any $e_r \in (\underline{e}, \overline{e})$, there exists an $\epsilon > 0$ such that all rules $e' \in (e_r - \epsilon, e_r + \epsilon)$ are enforceable. Therefore any $e_r \in (\underline{e}, \overline{e})$ is strongly enforceable.

In figure 1, $\underline{u}^I_{SF}(e) < y < \overline{u}^I_{SF}(e)$ for the entire range of result with multiple equilibria, though for some parameterizations we have simulated this is not the case and y intersects one of these curves. As demonstrated in the proposition, this does not change the qualitative conclusions, as there is still a range of strongly enforceable election results, just not the entire range with multiple equilibria. While we have not been able to prove that it is impossible for y to lie outside of $(\underline{u}^I_{SF}(\underline{\underline{e}}), \overline{u}^I_{SF}(\overline{\overline{e}}))$, which would imply there are no strongly enforceable electoral rules, exploring a wide range of parameterizations did not yield a case where this occurs.

Proof of proposition 4

For part i, implicitly differentiating the equilibrium condition gives that when there is a unique equilibrium $\hat{\theta}(e)$ for all e, $\frac{\partial \hat{\theta}}{\partial e}$ is continuous and strictly decreasing in c, so u^I_{SF} is strictly increasing in c, and hence the e solving $u_{SF}(e) = y$ must be strictly decreasing in c.

For part ii, first define $\overline{\hat{\theta}}(e; c)$ as the highest equilibrium threshold rule at e and c, $\underline{\hat{\theta}}(e; c)$ as the lowest equilibrium threshold rule, and now we write $\underline{u}^I_{SF}(e; c)$ and $\overline{u}^I_{SF}(e; c)$ with c arguments as well. Since we restrict attention to equilibria where G is strictly decreasing in θ, the implicit function theorem implies that $\overline{\hat{\theta}}(e; c)$ and $\underline{\hat{\theta}}(e; c)$ are continuous in e and c. By the continuity of the incumbent utility function in ρ, $\underline{u}^I_{SF}(e; c)$ is continuous in e and c if and only if $\underline{\hat{\theta}}(e; c)$ is continuous in e and c, and $\overline{u}^I_{SF}(e; c)$ is continuous in e and c if and only if $\overline{\hat{\theta}}(e; c)$ is continuous in e and c. So, $\underline{u}^I_{SF}(e_r; c_0) < y^I < \overline{u}^I_{SF}(e_r; c_0)$ implies $\underline{u}^I_{SF}(e_r; c_0 \pm \epsilon) < y^I < \overline{u}^I_{SF}(e_r; c_0 \pm \epsilon)$ for sufficiently small ϵ. Hence e_r is strongly enforceable for $c_0 \pm \epsilon$ for sufficiently small ϵ. ∎

Proof of proposition 5

Consider the equilibrium condition as $\tau_e \to \infty$. First, σ_{RHS} approaches 0, indicating that the left-hand side of the equilibrium condition approaches a step function equal to 0 for $\hat{\theta}(e) < \mu_{RHS}(e)$ and 1 for $\hat{\theta}(e) > \mu_{RHS}(e)$. Write $\mu_{RHS}(e)$ as:

$$\frac{\tau_e e + \tau_0 \mu_0}{\tau_0 + \tau_e} = \frac{\tau_0 + \tau_e + \tau_\theta}{\tau_0 + \tau_e} (\tau_\theta)^{-1/2} \Phi^{-1}(\rho^*)$$

And hence:

$$\lim_{\tau_e \to \infty} \mu_{RHS}(e) = e + \tau_\theta^{-1/2} \Phi^{-1}(\rho^*)$$

Which is linearly increasing in e. This implies that as $\tau_e \to \infty$, the equilibrium threshold is unique at $\hat{\theta}(e) = \overline{\theta}$ for $e < \underline{\theta} - \tau_\theta^{-1/2}\Phi^{-1}(\rho^*)$ and unique at $\hat{\theta}(e) = \underline{\theta}$ for $e > \overline{\theta} - \tau_\theta^{-1/2}\Phi^{-1}(\rho^*)$. For e in between these bounds, there are intersections at both $\hat{\theta}(e) = \underline{\theta}$ and $\hat{\theta}(e) = \overline{\theta}$.

If the election is fully informative, than the marginal citizen has no additional private information about ω and hence the left-hand side of the equilibrium condition is the probability that the protest will induce the incumbent to step down. So if the citizens select $\hat{\theta}(e) = \underline{\theta}$ then no citizens without a dominant strategy protest and the incumbent stays in power with probability 1, and if $\hat{\theta}(e) = \overline{\theta}$ all citizens without a dominant strategy to protest would do so and the incumbent steps down prior to protest, hence the moderates do not protest on the equilibrium path.

Proof of propositions 6-7

With the addition of the monitoring report the interim mean belief for citizens before observing their regime sentiment is:

$$\overline{\mu}(e,m) = \frac{\tau_0\mu_0 + (\tau_e+\tau_m)(e-x) + \tau_m(m-x)}{\tau_0+\tau_e+\tau_m}$$

Both are linearly increasing in e and the belief with the monitoring report is linearly decreasing in m.

Lemma 1. *i. We can write the equilibrium threshold rule as the mean interim popularity: $\hat{\theta}(\overline{\mu}(e))$ or $\hat{\theta}(\overline{\mu}(e,m))$, and*
ii. In both the baseline model and extension, all of the results referring to the election result can be rewritten with respect to the interim mean popularity

Proof The citizen posterior belief about the incumbent popularity with the monitoring report and their regime sentiment has mean:

$$\overline{\overline{\mu}}(e,m,\theta_j) \equiv \frac{(\tau_0+\tau_e+\tau_m)\overline{\mu}(e,m)+\tau_\theta\theta_j}{\tau_0+\tau_e+\tau_m+\tau_\theta} = \frac{\tau_0\mu_0 + (\tau_e+\tau_m)(e-x) - \tau_m(m-x) + \tau_\theta\theta_j}{\tau_0+\tau_e+\tau_m+\tau_\theta}$$

and precision $\tau_0+\tau_e+\tau_m+\tau_\theta$. Replacing $\overline{\mu}(e,\theta_j)$ with $\overline{\overline{\mu}}(e,m,\theta_j)$ and $\tau_0+\tau_e+\tau_\theta$ with $\tau_0+\tau_e+\tau_m+\tau_\theta$ in equation 2 gives an analogous equilibrium condition for $\hat{\theta}(e,m)$, which is decreasing in m in the unique equilibrium case (and the high and low protest equilibria with multiple equilibria) by an analogous argument. For part ii, it is clear from definition that $\overline{\mu}(e)$ is an invertible function of the election result e. So, as long as there is a unique equilibrium for all e there will be a unique equilibrium for all $\overline{\mu}(e)$ and a one-to-one correspondence between these descriptions of the equilibrium threshold.[2] So all comparative statics with

[2]The uniqueness caveat is only necessary because when there are multiple equilibria the mapping from the election result to equilibrium thresholds is not a function.

respect to e can be equivalently derived with respect to $\overline{\mu}(e)$ in the baseline model and $\overline{\mu}(e, m)$ with the monitoring report. $\blacksquare$

Propositions 6-7 follow from lemma 1 and the fact that $\overline{\mu}(e, m)$ is linearly decreasing in m.

References

Greene, William H. 2008. *Econometric Analysis, Sixth Edition*. Prentice Hall.

LaGatta, Tom. 2013. "Continuous disintegrations of Gaussian processes." *Theory of Probability & Its Applications* 57(1):151–162.

Schelling, Thomas C. 1960. *The Strategy of Conflict*. Harvard University Press.